

# FORWARD FITTING TO QUOTES OF AMERICAN OPTIONS

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## Abstract

*The paper discusses different approaches to calibrate forward components such as dividends or borrow rates based on a given set of American option quotes. While deducing forward prices from European option prices is straight forward due to the put-call parity, it is not directly obvious how to proceed if only American option prices are available. We describe the complexity of this problem and we analyze a widely used practitioners approach using implied volatilities. An illustrative example is used to demonstrate why this ad-hoc approach exhibits undesirable properties and can lead to unreasonable results. Finally, we propose a different algorithm to circumvent these problems.*

## 1 Forward Fitting

Across all assets classes, forward curves are a central element of derivative markets and a common exercise within controlling and trading departments of all market participants is to construct an internal theoretical forward curve  $T \mapsto F^t(T)$  for all maturities  $T > 0$  for an arbitrary tradable asset, where  $F^t(T)$  denotes the theoretical forward price for delivery at maturity  $T$ . Relying on well-known economic replication principles, market participants typically represent a theoretical forward curve  $F^t$  using different components  $x_1, \dots, x_k$ , such as a funding curve, storage costs, some form of convenience yield, a borrow curve and - within equity markets - dividends. Let us assume, that all but one of the relevant forward components for the construction of the theoretical

forward curve are directly observable in the market and known. The remaining unknown forward component is denoted by  $x$  and the representation of the theoretical forward is given by:

$$F^t(T) = f(T, x(T))$$

The latent factor  $x$  can be thought of either an illiquid or unobservable forward component or simply a fudge factor. We assume that for a fixed maturity  $T$ , the theoretical forward price  $F^t(T)$  depends strictly monotonically on  $x_T = x(T)$ , which allows us to deduce the value  $x_T$  given the value  $F^t(T)$ . In the following subsections, we will explain how values for the unknown latent factor curve can be obtained for different categories of market information.

### 1.1 Forward quotes

In markets, where quotes for physically or cash settled forward contracts, ie. bid and ask prices  $F^b(T_i)$  and  $F^a(T_i)$ , are directly observable in the market for a set of expiries  $T_1, \dots, T_n$ , we can directly derive constraints for the values  $x_{T_i}$  of our latent factor from the market conformity condition for the theoretical forward price:

$$\forall i \in \{1, \dots, n\} : F^b(T_i) \leq F^t(T_i) \leq F^a(T_i)$$

Obviously, theoretical forwards - and consequently also the values  $x_{T_i}$  of the latent factor - are not uniquely defined in the case of non-trivial bid-ask quotes. Furthermore, if no other market information about forward prices between and outside of quoted expiries is available, then the theoretical forward curve  $F^t(T)$  is subject to the individual shape of the used intertemporal inter- and extrapolation function.

## 1.2 European option quotes

In markets where no direct forward quotes, but European option quotes are available, the situation is very similar. For a given strike  $K$  and maturity  $T$ , we denote the European call option price for this strike and maturity by  $C(K, T)$  and the corresponding European put option price by  $P(K, T)$ . In the absence of arbitrage, the put-call parity implies for all strikes  $K$ , independently of the model used to price the options, the following equality:

$$e^{rT} \cdot (C(K, T) - P(K, T)) + K = F^t(T) \quad (1)$$

where  $r$  denotes the continuously compounded yield for maturity  $T$  for an appropriate discount curve. If for a quoted strike  $K$  in maturity  $T = T_i$ , we denote by  $C^b$ ,  $P^b$  and  $C^a$ ,  $P^a$  the bid and ask prices for the corresponding call and put option, then, it is straightforward to derive the following strike-specific forward quote from equation (1):

$$\begin{aligned} F^b(K, T_i) &:= e^{rT} \cdot (C^b - P^a) + K \\ F^a(K, T_i) &:= e^{rT} \cdot (C^a - P^b) + K \end{aligned}$$

If different strikes  $K_j$ ,  $j = 1, \dots, n$  are available for expiry  $T_i$ , then all quotes can be used to derive a sharper constraint for the theoretical forward price  $F^t(T_i)$ . Defining

$$\begin{aligned} F^b(T_i) &= \max_K F^b(K, T_i) \\ F^a(T_i) &= \min_K F^a(K, T_i) \end{aligned}$$

we obtain the following inequality that must be fulfilled by the theoretical forward price  $F^t$ :

$$F^b(T_i) \leq F^t(T) \leq F^a(T_i)$$

As a consequence, forward fitting based on European option quotes is comparable to the situation where direct quotes for the forward are observable in the market.

## 1.3 European implied volatility quotes

Among option traders and market makers, it is very common to consider implied volatilities instead of option prices while calibrating the relevant market input parameters. More precisely, market participants use an option pricing model, such as e.g. the Black-Scholes model to calculate the corresponding implied volatilities  $\sigma(K, T)$  from the given set of bid-ask quotes for European options.

As a result, we obtain bid and ask volatilities from the given quotes for the call and put option with a given strike  $K$  and given expiry  $T$ :

$$\begin{aligned} \sigma_P^b(K, T) &\leq \sigma_P^a(K, T) \\ \sigma_C^b(K, T) &\leq \sigma_C^a(K, T) \end{aligned}$$

The obtained bid-ask values for the implied volatilities depend critically on the theoretical forward curve  $F^t(T)$  used for the implied volatility calculation<sup>1</sup>. For a fixed strike  $K$  and maturity  $T$ , we obtain the following result:

$$\begin{aligned} F_K^b(K, T) \leq F^t(T) \leq F_K^a(K, T) \\ \Leftrightarrow \\ [\sigma_P^b, \sigma_P^a] \cap [\sigma_C^b, \sigma_C^a] \neq \emptyset \end{aligned}$$

In other words, the theoretical forward price is consistent with the bid-ask forward quote implied from the European option quotes if and only if, the deduced implied volatility quotes for calls and puts have a non-empty intersection. This result is illustrated in figure (1).

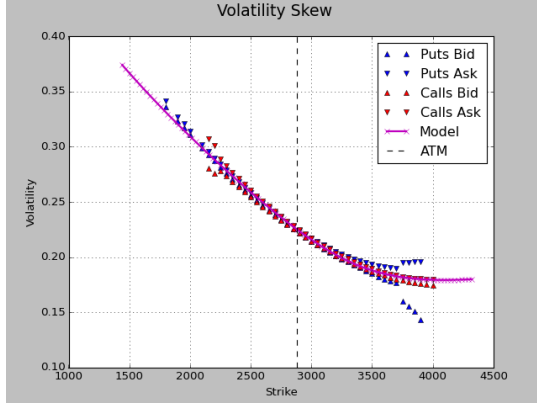
## 1.4 American Option Quotes

In markets where only American option quotes are available, the situation is slightly different as American call and put option prices do not feature a put-call parity comparable to equation (1). Instead, the assumption of absence of arbitrage implies only the following inequalities:

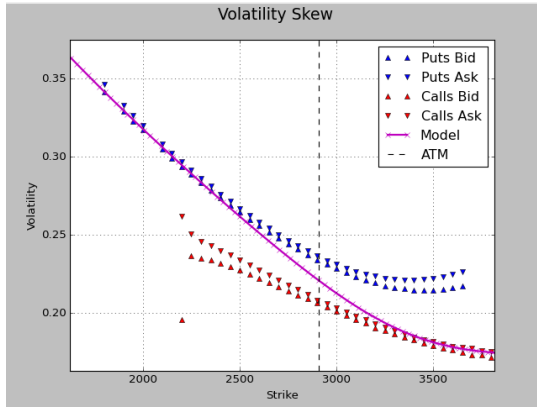
$$S_0 - K \leq c(K, T) - p(K, T) \leq F^t(T) - Ke^{rT}$$

where we denote the price of an American call option with strike  $K$  and maturity  $T$  by  $c(K, T)$  and the respective price of the put option by  $p(K, T)$ . As before,  $r$  denotes the continuously compounded yield for maturity  $T$  for an appropriate discount curve. To prove the first inequality, we consider a portfolio of long an American put, short an American call and long a share of the underlying stock. If the American call is exercised we deliver the stock and get an amount of  $K$  in cash. In this case, at maturity we have cash of at least  $K$ . If the option has not been exercised early and  $S_T > K$ , we deliver the stock and receive  $K$  in cash and finally if  $S_T < K$  we exercise the put and receive  $K$  in cash. Therefore we have a payoff of at least  $K$  in

<sup>1</sup>Remark: The discount curve plays another important role in the implied volatility calculation, but it is straightforward to calibrate an appropriate discount curve from a given set of option quotes and therefore we stick to our assumption that the discount factors are known.



(a) The quotes do overlap, the theoretical forward is consistent with the option quotes.



(b) The quotes have an empty intersection, the theoretical forward is not consistent with the option quotes.

Figure 1: Implied volatility quotes for put options (blue) and call options (red) for different values of the theoretical forward used to calculate the implied volatilities.

all situations which leads to the first inequality and a similar argument leads to the second inequality and results in a lower bound for the theoretical forward price:

$$F^t(T) \geq c(K, T) - p(K, T) + Ke^{rT}$$

Unfortunately this inequality is not sharp enough to calibrate the latent factor  $x_T$  in a comparable way as described in the case of European options. A practical approach - seemingly best-practice amongst traders - computes the implied volatilities from the American option quotes and then, in a next step uses these implied volatilities to compute the respective European option prices and finally computes quotes for the forward price using equation 1. As the American implied volatilities depend on the assumed forward price, this proce-

cedure is repeated using the newly calibrated forward price for again computing the American implied volatilities and European option prices until convergence in the implied forward price is achieved. For a fixed strike  $K$ , fixed maturity  $T$  and - for the ease of presentation - assuming zero bid-ask spread, this algorithm can be summarized as follows:

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**Algorithm 1** Implied Forward Fitting via European put-call Parity

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Initialize  $k = 0, x_0, F_0 = F(x_0)$ 
while  $k < k_{\max}$  and  $\|x_k - x_{k-1}\| > \varepsilon$  do
  compute  $\sigma^c(F_k)$  and  $\sigma^p(F_k)$ 
  compute  $C(\sigma^c(F_k))$  and  $P(\sigma^p(F_k))$ 
  compute  $F_{k+1} = e^{rT}(C - P) + K$ 
  determine  $x_{k+1}$  s.t.  $F(x_{k+1}) = F_{k+1}$ 
   $k \leftarrow k + 1$ 
end while

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This approach is equivalent to treating the American implied volatilities as European implied volatilities and adjusting the theoretical forward price used for the implied volatility calculation such that the intersection of implied volatility quotes of call and put options is non-empty - comparable to the methodology described for European options and illustrated through figures (1a) and (1b). With other words, this algorithm tacitly relies on the assumption that the implied volatility of an American call and put option is somehow related to the implied volatility of the respective European counterparts which in turn would imply that the implied volatility of American calls and puts with same strike should have very similar implied volatilities. Note that this assumption may be valid in cases where early exercise is not very likely due to low rates and dividends but in general this assumption is not fulfilled.

To clarify that in general situations this intuition leads to wrong results, let us consider the following example: We assume that the underlying dynamic is given by a simple diffusion model with zero drift and a deterministic time dependent volatility function:

$$dS_t = S_t \sigma(t) dW_t, S_0 = 100 \quad (2)$$

Furthermore, we assume a proportional dividend of  $\delta = 5\%$  at  $t = 0.5$ :

$$S_t = (1 - \delta) \cdot S_{t-} \text{ for } t = 0.5$$

The given model implies a time dependent but strike independent Black-Scholes implied volatility surface for European options. The volatil-

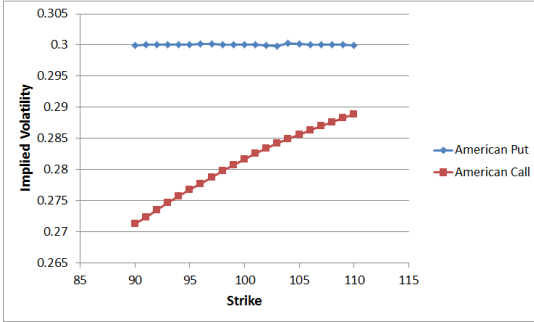


Figure 2: Implied volatilities for American call and put option prices as a function of the strike base on model (2).

ity function  $\sigma(t)$  is chosen such that the implied volatilities exhibit a linear term structure starting with a value of 20% at valuation date  $t = 0$  and rising to 30% implied volatility at one year ( $t = 1.0$ ). Within the given model, we calculate American call and put option prices for options with different strikes maturing in one year. These option prices are calculated using the model consistent forward price  $F(T) = 95$  for the one year maturity and these prices will now be used as input for algorithm 1. Figure 2 displays the obtained implied volatilities. and we observe significant differences in call and put implied volatilities although the theoretical forward used to calculate the implied volatilities is equal to the model forward. We clearly see the effect that the implied volatility of the American call is lower than the implied volatility of the respective put due to the probability of early exercise and the term structure of volatility.

If we apply algorithm 1 in order to calibrate the dividend yield based on the model prices, the resulting calibrated dividend yield  $\delta^*$  equals approximately 6%. Table 1 shows the iteration steps of the algorithm starting with a zero dividend yield. It is quite obvious that convergence is reached very fast but unfortunately not to the correct value.

| #iter | dividend |
|-------|----------|
| 0     | 0%       |
| 1     | 4.80%    |
| 2     | 5.64%    |
| 3     | 5.86%    |
| 4     | 5.92%    |

Table 1: Iterates of Algorithm 1 applied to model prices generated with model (2).

Figure 3 visualizes the impact of the assumed discrete dividend  $\delta$  on the corresponding theoret-

ical implied volatility of a American call option at different strike levels  $K$ . A larger dividend  $\delta$  and a smaller strike  $K$  imply a higher probability for early exercise and therefore a higher deviation from the terminal European implied volatility at maturity of 30%. The American put implied volatility instead does not react to a change in strike or dividend yield and is identical to its European counterpart with value 30%. Consequently, if the implied volatilities of European options exhibit a non-trivial termstructure and if the underlying pays a non-zero dividend, then the intuitive trader's approach of matching implied volatilities also in the case of American options will lead to wrong results.

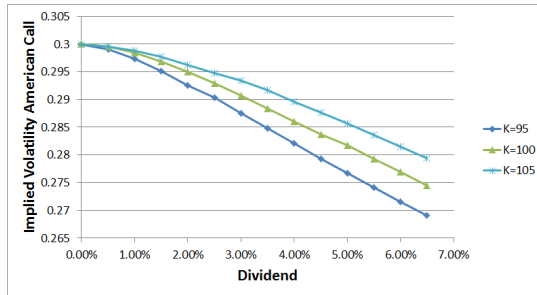


Figure 3: Implied volatilities for American call option for different strikes as a function of the dividend yield for model (2).

## 1.5 A practical alternative

The key misconception of the above approach is the use of implied volatilities as a criteria for accuracy of the forward price in the case of American option quotes because American implied volatilities are not directly linked to a single maturity. Therefore we propose to fit the forward price using the option prices directly.

Let us assume that we have a parametrization for the *European* implied volatility surface depending on some parameters  $\zeta \in \mathbb{R}^n$  where we get for each strike  $K$  and maturity  $T$  a European implied volatility  $\sigma(\zeta, K, T)$ . For a given expiry  $T$ , we have market prices  $\{c_i\}$  and  $\{p_i\}$  of American call and put options for a set of strikes  $\{K_i\}$ . We further denote the price of an American call option with strike  $K_i$  and maturity  $T$  using the local volatility model<sup>2</sup> with implied volatility  $\sigma(\zeta, \dots)$

<sup>2</sup>Although one could use any model we propose to use the local volatility model since the pricing of American options in this model is quite robust and efficient in contrast to other more realistic models such as stochastic volatility models. Note that

and theoretical forward price  $F^t(T, x)$  by  $c_i^t(\zeta, x)$  (and  $p_i^t(\zeta, x)$  for the put respectively). Let us assume that for  $x_T := x^0$  fixed we find a  $\zeta^0$  minimizing the least square sum of differences between market and model prices:

$$f(\zeta, x) = \sum_i (c_i - c_i^t(\zeta, x))^2 + (p_i - p_i^t(\zeta, x))^2$$

We now distinguish three different cases:

**Case 1:** Let us assume that for all  $i$

$$c_i > c_i^t(\zeta^0, x^0) \text{ and } p_i < p_i^t(\zeta^0, x^0)$$

or

$$c_i < c_i^t(\zeta^0, x^0) \text{ and } p_i > p_i^t(\zeta^0, x^0).$$

Due to the monotonicity of  $c$  and  $p$  w.r.t  $F$ , we know that we can reduce each summand of the least square error and therefore the complete sum by adjusting  $x^0$  to  $x^1$  so that the resulting theoretical forward  $F^t(T, x^1)$  is smaller than  $F^t(T, x^0)$ . If the reverse of these inequalities holds true, we adjust  $x^0$  to  $x^1$  such that the forward is increased. In both cases we are decreasing each error term of the cost function which is leading to an overall reduction of the cost. Therefore the new value  $x^1$  exhibits a reduced overall fitting error. Based on the new value  $x^1$ , the volatility parametrization is calibrated leading to a new value  $\zeta^1(x^1)$ .

**Case 2:** Let us assume that for all  $i$

$$c_i > c_i^{theo}(\zeta^0, x^0) \text{ and } p_i > p_i^{theo}(\zeta^0, x^0)$$

or the reverse holds true. In this situation, it is possible to reduce the cost function by reducing the overall level of implied volatility, contradicting the optimality of  $\zeta^0$ . Therefore this case cannot occur (up to rounding errors).

**Case 3:** Let us assume that the above inequalities do not hold for all  $i$ . In this case, it is difficult to determine whether the difference between the theoretical and the observed prices is due to the fit of the implied volatility surface or due to the used theoretical forward. If this case occurs, there is not enough information to proceed with the forward fitting and we stop the algorithm<sup>3</sup>.

For the ease of presentation, we provide the pseudo-code of the fitting algorithm for the single

although the choice of model influences the estimated forward, to our experience the differences are quite small which motivates the use of the local volatility model.

<sup>3</sup>If we assume, the the chosen volatility parametrization is flexibel enough to fit market implied volatilities, this case is fortunately of negligible importance. From our experience it is sufficient to handle this case by applying one single iteration of the optimization w.r.t.  $x$  in order to reduce the overall least square error.

strike case, where the cost function  $f$  consists of only one summand:

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**Algorithm 2** Implied Forward Fitting by nested Volatility and Forward Fitting

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Initialize  $k = 0, x^0$ 
while  $k < k_{\max}$  and  $\|x^k - x^{k-1}\| > \varepsilon$  do
   $\zeta_k = \arg \min_{\zeta} f(\zeta, x^k)$  (Fit of volatility)
  if  $(c^t > c \text{ and } p^t > p)$  or
      $(c^t < c \text{ and } p^t < p)$  then
    stop
  else
     $x^k = \arg \min_x f(\zeta_k, x)$ 
  end if
   $k \leftarrow k + 1$ 
end while

```

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Obviously, this algorithm can be easily extended to handle non-trivial bid-ask quotes, multiple strikes and multiple expiries.

## 1.6 Concluding Remarks

We have analysed a common practitioners approach to calibrate forwards in the case of American options and demonstrated by means of a simple and illustrative example why this method can lead to incorrect results. We proposed a method based on a nested calibration of implied volatilities and forward values where the cost function includes prices rather than volatilities. This approach circumvents the aforementioned problems of the practitioners approach.