A Co-Integrated Stochastic Volatility Model for Energy Derivatives

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Content

Example - Gas Prices and Location Spread

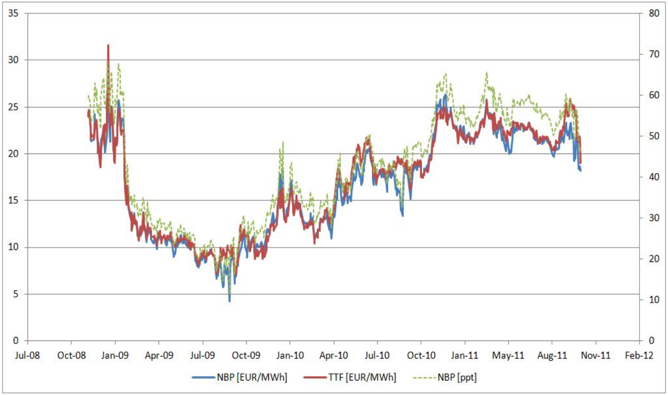

Figure : UK (NBP) and Netherland (TTF) gas spot prices. Left axis [EUR/MWh], right axis pence per therm [ppt].

- Very strong co-movement over large periods of time, due to interconnected markets.
- Possible applications: Valuation of transmission capacity (location spread optionality); Sourcing of load schedules.

Example - Oil Products and Crack Spread

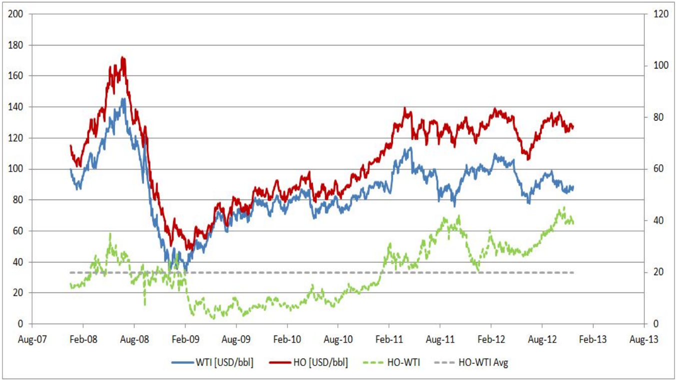

Figure : Crude oil (WTI) and Heating oil nearest future prices. Left axis [USD/bbl], right axis differences (spreads).

- Loose co-movement with periods of deviation.
- Possible applications: Crack spread option valuation; Volatility surface generation.

Example - WTI Crude Oil and Time Spread

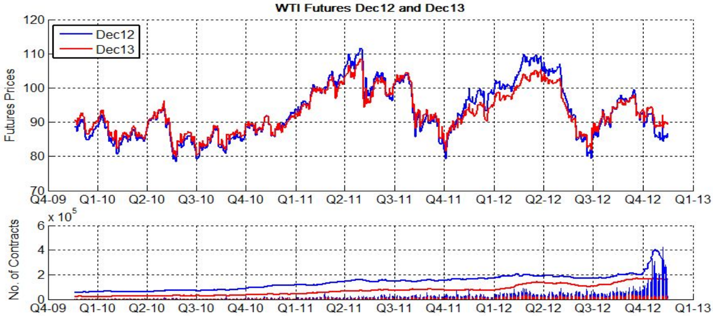

Figure : Upper: Dec12 and Dec13 WTI Crude Oil futures prices. Lower: Traded volumes and open interest.

- Strong co-movement when time-to-maturity is large, but decoupling as maturities come closer.
- Possible applications: Time spread option valuation; Valuation of long-term contracts/derivatives.

Model Requirements

- Market Structure:
  - Two or more underlying that show co-dependence
  - Dependence structure with "Leader" and "Follower" (e.g. due to liquidity)
- Viable Model dynamics:
  - Model allows to represent strongly co-dependend assets (without running into extreme parameter constellations)
  - Some parameters are statistically stable over time (physical measure), hence could be estimated from historical time series.
  - Stochastic volatility (model generates market-realistic implied volatility surfaces)
- Efficient valuation of contingent claims (e.g. in view of calibration):
  - Plain vanilla options
  - Spread dependent claims

Modelling the Co-Movement

Guided by the observable linear dependency of differenced futures prices

$$dG_t \approx b dF_t$$

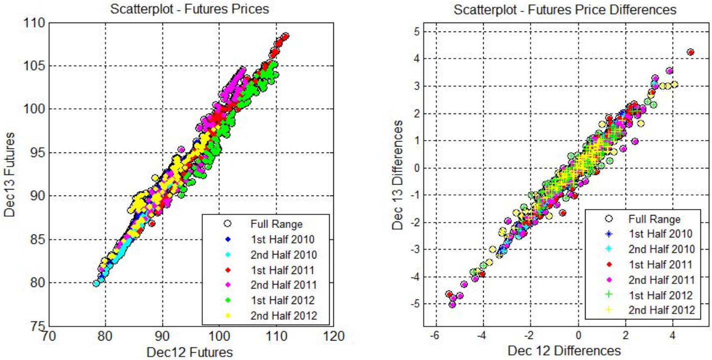

Figure : Time/regime dependency of absolute prices. Price differences show presence of volatility clusters.

Modelling the Co-Movement

We propose the dynamics

$$dG_t = bdF_t + dZ_t$$

where

F	:	is the leading instrument (with a dynamics to be specified later)
G	:	is the following instrument
Z_t	:	driven by an Ornstein-Uhlenbeck process, i.e. $dZ_t = -\kappa(Z_t - \theta)dt + \eta dW_t^Z$

Integration yields

$$G_t = \underbrace{G_0 - bF_0 - Z_0}_{=:a} + bF_t + Z_t.$$

- The relation of F_t and G_t in this model deviates from the statistical definition of co-integrated time series, since we do not require the residuum process Z_t to be in a stationary state.
- It can be shown that the residuum process is ergodic, i.e. converges to stationarity, if the usual OU conditions on the process parameters are satisfied.

Statistical Analysis

Estimating Z_0 and b in

$$G_{t+\Delta t} - G_t = b(F_{t+\Delta t} - F_t) + Z_{t+\Delta t} - Z_t$$

by least-squares-minimizing the ϵ_t's in

$$Z_{t+\Delta t} = \underbrace{(1-\kappa\Delta t)}_{\beta} Z_t + \underbrace{\kappa\theta\Delta t}_{\alpha} + \underbrace{\eta(W^Z_{t+\Delta t} - W^Z_t)}_{\epsilon}.$$

Expirations of the December-contracts in November suggest to consider the following different regimes:

Regime		Z_0	b	κ	θ
(2010)	20-Nov-2009 – 19-Nov-2010	0.0292	0.9596	4.5050	-1.2355
(2011)	20-Nov-2010 – 18-Nov-2011	0.8855	0.8945	3.5253	0.5753
(2012)	19-Nov-2011 – 16-Nov-2012	0.3798	0.8037	3.5880	4.0033

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Statistical Analysis

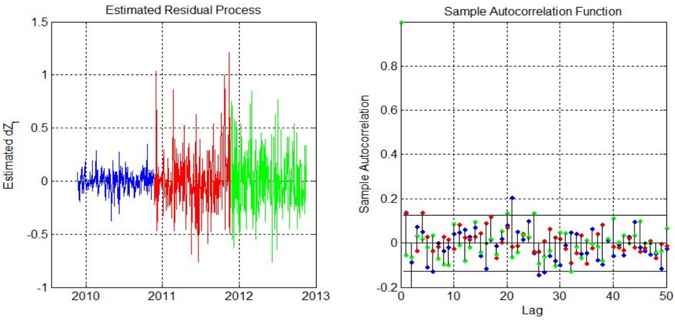


Figure : Least-Squares estimation of the residual process dZ_t and its sample auto-correlation function.

Tests for unit roots of the estimated dZ_t process. ADF and Variance Ratio have "unit-root" as null hypothesis, KPSS and LMC have "(trend) stationary process" as null hypothesis.

Regime	p-values for ...	ADF	Var'ratio	KPSS	LMC
(2010)		$< 0.1\%$	$< 0.1\%$	$> 10\%$	$> 10\%$
(2011)		$< 0.1\%$	0.12%	$> 10\%$	3.35%
(2012)		$< 0.1\%$	$< 0.1\%$	$> 10\%$	$> 10\%$

Interpretation of Cost-of-Carry

Let the maturities of F and G be T_F and T_G, respectively, where $T_F < T_G$. The difference

$$Y_t := G_t - F_t$$

is the price (as seen at time t) for carrying the physical underlying from time T_F to T_G.

What is the dynamics of this cost-of-carry process?

Integrating dG_t yields $G_t = \underbrace{(G_0 - bF_0 - Z_0)}_{=:a} + bF_t + Z_t$, hence

$$Y_t = a + (b-1)F_t + Z_t.$$

Interpretation:

a, θ	Fixed cost
$(b-1)$	Price level dependent "convenience"
Z_t	Mean reverting noise process

Backwardation relates to $\theta - Z_0 < 0$, $b < 1$ and contango to $\theta - Z_0 > 0$, $b > 1$.

F-Dynamics

F_t is supposed to be the liquidly traded instrument, possibly with traded derivatives on F. Hence, we want to achieve:

- market-realistic implied volatility surfaces,
- avoid unrealistic volatility dynamics,
- efficient valuation of market instruments for calibration.

F follows a stochastic volatility model of Heston's type. Under the physical measure, the dynamics is given by

$$\begin{aligned} \frac{dF_t}{F_t} &= \mu_F dt + \sqrt{V_t} dW_t^{(1)} + \sqrt{U_t} dW_t^{(2)} \\ dV_t &= -\zeta(V_t - \nu)dt + \sigma\sqrt{V_t} dW_t^V \\ U_t &= \gamma(Z_t - \theta)^2 \end{aligned}$$

with (as specified before)

$$dZ_t = -\kappa(Z_t - \theta)dt + \eta dW_t^Z, \quad dG_t = b dF_t + dZ_t,$$

and

$$dW_t^{(1)} dW_t^V = \rho_V, \; dW_t^{(2)} dW_t^Z = \rho_Z,$$
$$dW_t^{(1)} dW_t^{(2)} = dW_t^{(1)} dW_t^Z = dW_t^{(2)} dW_t^V = 0$$

Feedback Volatility U

- Empirically, one observes that deviations from the "normal" spread level can go along with increased volatility of the individual underlying.
- Possible explanation: Shocks in the short-term influence the short-end prices and volatilities, but less the long-term dated futures. As a consequence, this widens the spread.

Hence, we include an additional source of randomness in F via the deviation of Z to θ:

$$U_t = \gamma(Z_t - \theta)^2$$

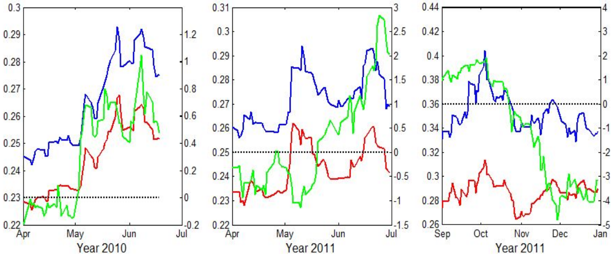


Figure : Implied volatilities of the Dec12 (blue) and Dec13 (red) futures (left axis). The deviation $(Z_t - \theta)^2$ (black; right axis).

Comparison to Other Models

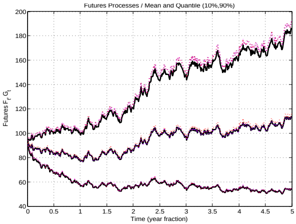
- Multi-factor correlation models (GBM and extensions)
  - Suffer from unexplainable/unrealistic extreme correlations to capture non-diffusing spread behavior
  - Almost impossible to avoid spreading underlying in the long-term
  - Efficient spread option pricing usually requires approximations (e.g. Margrabe formula, Kirk's approximation)
- Dempster, Medova and Tang (2006): *Long term spread option valuation and hedging.*
  Similar approach, but developed on spot prices and spot spreads.
  Differences of our approach to theirs:
  - Model for the discrete futures curve by means of strings (market model approach)
  - Spread depending volatility ("feedback" volatility U_t)
  - Incorporation of stochastic volatility
- Duan and Pliska (2004): *Option Valuation with Co-integrated Asset Prices*
  GARCH type model in discrete time; equilibrium based valuation.
- Approaches of the type "stochastic spot convenience yield".

Comparison with a Multi-Factor GBM

A typical problem with multi-factor models is:

- The individual futures processes F_t and G_t show a very similar behavior for co-integration and correlation models.
- But to restrain the spread $G_t - F_t$ the correlation has to be set unrealistically high in the multi-factor model.

Simulations of co-integration and correlation models (under the risk-neutral measure).



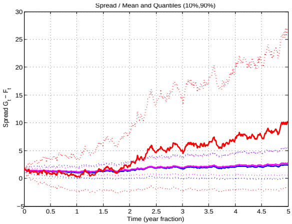


Figure : F (black) and G (blue=co-integration, red=correlation 0.99, magenta=correlation 0.9999).

Equivalent Martingale Measure

We assume that both futures F and G are tradable

Replicating a (short) position in G by a position in F and physical carry.

Time	Cash-flows	
t	0	
T_F	$-F_t - Y_{T_F}$	Note: By definition, the carry-costs Y_{T_F} include
T_G	G_t	already the financing costs from T_F to T_G!

G forward price is given by

$$\mathbb{E}\left[D(T_G)\left\{(-F_t - Y_{T_F}) + G_t\right\} \mid \mathcal{F}_t\right] = 0,$$

where $D(T_G)$ is the deterministic discount factor for T_G. Rearranging,

$$\begin{aligned} G_t &= F_t + \mathbb{E}[Y_{T_F} \mid \mathcal{F}_t] = a + bF_t + \mathbb{E}[Z_{T_F} \mid \mathcal{F}_t] \\ &= a + \left(1 - e^{-\kappa(T_F - t)}\right)\theta + bF_t + e^{-\kappa(T_F - t)} Z_t. \end{aligned}$$

G_t is indeed a martingale, and the dynamics becomes

$$dG_t = b\,dF_t + \eta e^{-\kappa(T_F - t)} dW_t^Z.$$

Pricing Formula

Theorem – European option on F

The price of a European call option on F with time to maturity T and strike K is given by the pricing formula of a Double Heston call option,

$$\text{Call} = F_0 P_1 - e^{-rT} K P_2,$$

where (for $j = 1, 2$)

$$P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re}\Big[\frac{e^{-iu \log K} \varphi_j(u)}{iu} \Big] du$$

$$\varphi_j(u) = \exp\Big\{ iu \log F_0 + \sum_{k=1}^{2} \big(A_k^{(j)}(u) + B_k^{(j)}(u) v_k(0) \big) \Big\},$$

and the terms $A_k^{(j)}(u)$ and $B_k^{(j)}(u)$ are equivalent to the standard (Single-) Heston expressions.

Proof

1. Applying Itô's formula to U_t yields

$$dU_t = \gamma d\left((Z_t - \theta)^2\right)$$
$$= -2\kappa\big(U_t - \frac{\gamma\eta^2}{2\kappa}\big)dt + 2\sqrt{\gamma}\eta\sqrt{U_t}\ \mathrm{sign}(Z_t - \theta)dW_t^Z.$$

2. Define a modified process $\tilde{U}_t := (\tilde{Z}_t - \theta)^2$, where

$$d\tilde{Z}_t = -\kappa(\tilde{Z}_t - \theta)dt + \eta\mathrm{sign}(\tilde{Z}_t - \theta)dW_t^Z.$$

3. We know that $\tilde{U}_t = \gamma(\tilde{Z}_t - \theta)^2 \overset{d}{=} \gamma(Z_t - \theta)^2 = U_t$, and applying Itô's formula to $\tilde{U}_t$ yields

$$d\tilde{U}_t = -2\kappa\big(\tilde{U}_t - \frac{\gamma\eta^2}{2\kappa}\big)dt + 2\sqrt{\gamma}\eta\sqrt{\tilde{U}_t}dW_t^Z.$$

4. Accordingly, the log-transformed process $\tilde{F}$,

$$\frac{d\tilde{F}}{\tilde{F}} \quad = \quad \sqrt{V_t}dW_t^{(1)} + \sqrt{\tilde{U}}dW_t^{(2)},$$

is a double Heston process.

5. Since

$$\mathrm{Call} = \mathbb{E}[(F_T - K)^+ \mid \mathcal{F}_0] = \mathbb{E}[(\tilde{F}_T - K)^+ \mid \mathcal{F}_0],$$

the assertion follows.

Z_t-Parameter Influence on the Volatility Process $\tilde{U}_t$

From the derivation of the European F-option formula:

$$d\tilde{U}_t = -2\kappa\big(\tilde{U}_t - \frac{\gamma\eta^2}{2\kappa}\big)dt + 2\sqrt{\gamma}\eta\sqrt{\tilde{U}_t}dW_t^Z$$

- The feedback volatility influence is nil, if $U_0 = 0$ (which is equivalent to $Z_0 = \theta$), and $\gamma \approx 0$ or $\eta \approx 0$.
- Basically, one has control over four parameter combinations, κ, $\gamma\eta^2$, $(Z_0 - \theta)^2$ and ρ_Z.
- A change of mean-reversion speed implies a change in mean-reversion level.
  A change in vol-of-variance implies a change in mean-reversion level.
  A change in the mean-reversion level can have effect on both, mean-reversion speed and vol-of-variance.
- Changes in the $Z_0 - \theta$ relation are symmetric, i.e. independent of the sign.

Parameter Effects on F Implied Volatilities

Fix V_t variance process parameters

$$V_0 = (0.4)^2,\ \zeta = 3,\ \nu = (0.2)^2,\ \sigma = 0.5,\ \rho_V = 0.1$$

Setting for the U_t volatility process (blue curves below)

$$\kappa = 4.5,\ \theta = -1.2,\ \eta = 0.8,\ \gamma = 0.1,\ \rho_Z = 0,\ \text{ and } Z_0 = -1.2.$$

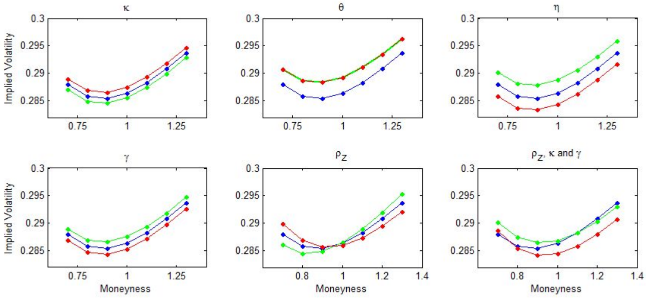


Figure : Plots 1-4: κ, θ, η and γ changed by $+10\%$ (green curves) and -10% (red curves), respectively. Plot 5: Correlation ρ_Z set to $+90\%$ (green) and -90% (red). Plot 6: $\rho_Z = -0.9$, and 2κ, 2γ (green), 0.5κ, 0.5γ (red).

Pricing Formula

Theorem – European option on G

The price of a European call option on G with maturity T and strike K is given by

$$\begin{aligned} \text{Call} &= \mathbb{E}\left[(G_T - K)\mathbb{1}_{\{G_T>K\}}\right] \\ &= a(T) - K + c(T)\mathbb{E}\left[\tilde{Z}_T\right] \\ &\quad + K\underbrace{\mathbb{E}\left[(1+c(T)\tilde{Z}_T)\mathbb{1}_{\{G_T<K\}}\right]}_{I} + b\underbrace{\mathbb{E}\left[\tilde{F}_T\mathbb{1}_{\{G_T>K\}}\right]}_{II} \end{aligned}$$

For the r.v. $(\tilde{Z}_T, \int_0^T(\tilde{Z}_s-\theta)^2 ds)$ we find solutions for I and II of the format

$$\begin{aligned} I &= \iint_{\mathbb{R}^2}(1+z)N\left(g_1(z,v)\right)\times \text{density}(z,v)dzdv \\ II &= \iint_{\mathbb{R}^2} e^{k_2(z-\theta)^2+k_3v+k_1}N\left(g_2(z,v)\right)\times \text{density}(z,v)dzdv \end{aligned}$$

The joint density is determined uniquely by its joint characteristic function and therefore given by Fourier inversion

$$\text{density}_{\{\tilde{Z}_T,\int_0^T(\tilde{Z}_s-\theta)^2ds\}}(z,v) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{-iuz-iwv}\phi_{z,v}(u,w)du\,dw,$$

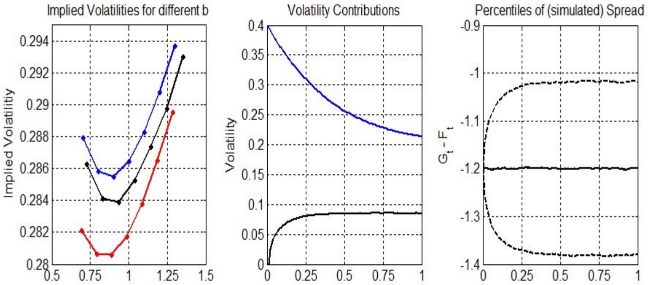
where the inner integration can be solved in closed-form.

Parameter Effects on G Implied Volatilities

Setting for the U_t volatility process (blue curves below)

$$\kappa = 4.5,\ \theta = 1.2,\ \eta = 0.8,\ \gamma = 0.1,\ \rho_Z = 0,\ Z_0 = \theta.$$

Left: The F implied volatility is given by the blue curve. The other curves are implied volatilities for options on G, whith parameter $b = 0.95$ (black) and $b = 1.0$ (red).
Middle: The average contributions of the processes $\sqrt{V_t}$ (blue) and $\sqrt{U_t}$ (black) from 100,000 simulations.
Right: 25%, 50% and 75% quantiles from 100,000 simulations for the spread process $G_t - F_t$.



Pricing Formula

Comparing the price of a European vanilla option on G with a European spread option on F and G with maturity T and strike K:

$$
\begin{aligned}
\mathbb{E}\left[(G_T - K)^+\right] &= \mathbb{E}\left[(a(T) + bF_T + c(T)Z_T - K)^+\right] \\
\mathbb{E}\left[(G_T - F_T - K)^+\right] &= \mathbb{E}\left[(a(T) + (b-1)F_T + c(T)Z_T - K)^+\right]
\end{aligned}
$$

European Options on F I

Theorem – European option on F

The price of a European call option on F with time to maturity T and strike K is given by the pricing formula of a Double Heston call,

$$\text{Call} = F_0 P_1 - e^{-rT} K P_2,$$

where (for $j = 1, 2$)

$$P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \text{Re}\Big[\frac{e^{-iu \log K} \varphi_j(u)}{iu} \Big] du$$

$$\varphi_j(u) = \exp\Big\{ iu \log F_0 + \sum_{k=1}^{2} \big(A_k^{(j)}(u) + B_k^{(j)}(u) v_k(0) \big) \Big\},$$

and

$$A_k^{(j)}(u) = \frac{\bar{\kappa}_k \bar{\theta}_k}{\bar{\sigma}_k^2} \Big((\bar{\kappa}_k - \bar{\rho}_k \bar{\sigma}_k ui - d_k^{(j)}) T - 2 \log\Big(\frac{1 - g_k^{(j)} e^{-d_k^{(j)} T}}{1 - g_k^{(j)}} \Big) \Big),$$

$$B_k^{(j)}(u) = \frac{\bar{\kappa}_k - \bar{\rho}_k ui - d_k^{(j)}}{\bar{\sigma}_k^2} \frac{1 - e^{-d_k^{(j)} T}}{1 - g_k^{(j)} e^{-d_k^{(j)} T}},$$

European Options on F II

$$d_k^{(j)} = \sqrt{(\bar{\kappa}_k - \bar{\sigma}_k\bar{\rho}_k(2-j) - \bar{\rho}_k\bar{\sigma}_k ui)^2 - \bar{\sigma}_k^2(ui - u^2)},$$

$$g_k^{(j)} = \frac{\bar{\kappa}_k - \bar{\sigma}_k\bar{\rho}_k(2-j) - \bar{\rho}_k\bar{\sigma}_k ui - d_j}{\bar{\kappa}_k - \bar{\sigma}_k\bar{\rho}_k(2-j) - \bar{\rho}_k\bar{\sigma}_k ui + d_j}.$$

The generic parameters above are given by

$$\begin{array}{llll} \bar{\kappa}_1 = \zeta & \bar{\theta}_1 = \nu & \bar{\sigma}_1 = \sigma & \bar{\rho}_1 = \rho_V \\ \bar{\kappa}_2 = 2\kappa & \bar{\theta}_2 = \frac{\gamma\eta^2}{2\kappa} & \bar{\sigma}_1 = 2\sqrt{\gamma}\eta & \bar{\rho}_2 = \rho_Z \end{array}$$